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Approximate Factorization as a High Order Splitting for the Implicit Incompressible Flow Equations

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We apply the method of approximate factorization to derive a second-order accurate splitting of the incompressible flow equations (Stokes or Navier–Stokes equations). This is novel because the method of approximate factorization was believed inapplicable to this type of equation system. We demonstrate the resulting splitting on a second-order Crank–Nicolson discretization and point out its intimate relationship to some existing second-order accurate splitting (projection) methods. Further, we use the approximate factorization method to derive entirely new splittings. We first develop a new generalized second-order accurate splitting which may be specialized to a variety of applications. We indicate its applications to finite-elements, "checkerboard-free" cell-centered discretizations, and ocean modeling. We then generalize the original splitting to an arbitrarily high-order scheme.

1. INTRODUCTION

We will be concerned with the problem of time discretization of the Navier–Stokes equations describing the flow of an incompressible fluid,

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \nu\nabla\cdot\nabla\mathbf{u}, \qquad \nabla\cdot\mathbf{u} = 0, \tag{1}$$

where $\mathbf{u}$ is the velocity, p is the pressure (divided by the density), and ν is the (constant) kinematic viscosity, together with the initial and boundary conditions. The initial conditions consist of a specified solenoidal velocity field and the usual boundary conditions involve specified velocities, $\mathbf{u} = \mathbf{w}$, on the boundary (for example, zero slip on solid boundaries); in such a case no boundary conditions for the pressure are needed. Spatial discretization is unspecified; for the moment we will assume a semi-discrete representation in which only the time is discretized.

The earliest methods such as MAC [1] or Chorin's method [2] employed a time discretization of these equations which operated in two steps, as

$$\text{(a)}\quad \frac{\tilde{\mathbf{u}} - \mathbf{u}^n}{\Delta t} + \mathbf{u}^n\cdot\nabla\mathbf{u}^* = \nu\nabla\cdot\nabla\mathbf{u}^*, \tag{2}$$

$$\text{(b)}\quad \frac{\mathbf{u}^{n+1} - \tilde{\mathbf{u}}}{\Delta t} = -\nabla p^*, \qquad \nabla\cdot\mathbf{u}^{n+1} = 0, \tag{3}$$

where Δt is the time step and the superscripts denote the time level ($\mathbf{u}^*$ represents $\mathbf{u}^n$ in MAC and $\tilde{\mathbf{u}}$ in Chorin's method; the pressure time level is unspecified in MAC and is taken to be the new time level by Chorin). Chorin [2] has given an elegant interpretation of the above procedure as a projection of the intermediate velocity field $\tilde{\mathbf{u}}$ onto a subspace of vector fields with zero divergence. One may also interpret Eqs. (2) and (3) as an operator splitting or fractional-step method, since by adding the two sets of equations to eliminate the intermediate velocity $\tilde{\mathbf{u}}$ one obtains a discrete approximation of the original equations [3] (strictly speaking, since the MAC method is explicit for the momentum equation it should not be so interpreted). Whichever interpretation is adopted this procedure has the great conceptual and practical advantage of decoupling the "pressure correction" from the momentum equation calculations.

Since that time many such decoupled methods for the incompressible flow equations have been developed. They have usually (but not always) adopted either the projection or the operator splitting point of view. Following Chorin, the projection method is at least partially inspired by the existence theory of the Navier–Stokes equations and therefore is grounded in the properties of Eqs. (1) in differential

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form. The operator splitting method originates as a purely numerical method which is general and not specific to a particular set of equations [3]. It should be emphasized that the practical end result when applied to the incompressible flow equations is the same in both cases, and the difference is only in the point of view and the personal preference of the practitioners. In this paper we take the operator splitting approach, and in particular we use the method of approximate factorization which is a particular form of the operator splitting method.

In the past there have been at least two additional but not independent issues in connection with this problem of decoupling: the issue of intermediate boundary conditions and the issue of time accuracy. Usually the problem is formulated in semidiscrete form as above, in which case the issue of the appropriate boundary conditions for the intermediate quantities is relevant and usually comes up when it is desired that the resulting method be second-order accurate. There is a subtle difference between the projection method and the operator splitting approach in this regard, in that it is conceptually easier to apply the operator splitting method to the fully-discrete form of the equations (in which the boundary conditions have already been applied) and therefore the question of intermediate boundary conditions never arises. As described later, this does not prevent the practitioners of the projection method from arriving at equivalent results! In the remainder of this section we will provide a brief and by no means exhaustive review of these issues as an introduction to our own approach which we show leads to a generalized framework for constructing new second- and higher-order accurate schemes. Henceforth, we will use the term splitting as a generic term for the decoupling, whether accomplished from the projection or the operator splitting point of view.

The above differencing of the equations is clearly first-order accurate in time and may require an excessively small time step Δt for an accurate integration in time, so that a second-order accurate discretization is usually of interest. Furthermore, in regions of high viscosity or small mesh size the viscous terms are a source of stiffness and one is therefore most frequently interested in an implicit form of the equations. The prototype for an implicit second-order accurate discretization of the Navier–Stokes equations is the system of Crank–Nicolson equations,

$$\begin{aligned} &\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\Delta t}+(\mathbf{u}\cdot\nabla\mathbf{u})^{n+1/2} \\ &\qquad =-\nabla p^{n+1/2}+\frac{1}{2}\nu\nabla\cdot\nabla(\mathbf{u}^{n+1}+\mathbf{u}^n), \\ &\nabla\cdot\mathbf{u}^{n+1}=0, \end{aligned} \tag{4}$$

where the precise discretization of the advection terms is left unspecified but is assumed to be explicit and time-centered to within second order. Individual methods often differ in the discretization of this term. This type of discretization, where the advection terms are explicit, is sometimes called semi-implicit [4]. Among the advantages of this discrete system is the fact that it requires precisely the same initial and boundary conditions as the original Navier–Stokes system, and in particular it does not require special boundary conditions for the pressure. The system may be solved directly or by various global iteration schemes; unfortunately, its computational solution is very expensive because all the variables, i.e., the velocity components and the pressure, are coupled and together they form a very large system. In order to alleviate this it was natural to consider the following splitting:

$$\text{(a)}\quad \frac{\tilde{\mathbf{u}}-\mathbf{u}^n}{\Delta t}+(\mathbf{u}\cdot\nabla\mathbf{u})^{n+1/2}=\frac{1}{2}\nu\nabla\cdot\nabla(\tilde{\mathbf{u}}+\mathbf{u}^n), \tag{5}$$

$$\begin{aligned} \text{(b)}\quad &\frac{\mathbf{u}^{n+1}-\tilde{\mathbf{u}}}{\Delta t}=-\nabla p^{n+1/2}, \\ &\nabla\cdot\mathbf{u}^{n+1}=0, \end{aligned} \tag{6}$$

where the first step represents a centered implicit equation for the effects of transport and viscosity and the second step is identical to the projection described above. The splitting is much more economical because it corresponds to the solution of several much smaller decoupled systems for the velocity components and the pressure, respectively. Again, this decoupling may be considered as an application of the projection method or as an operator splitting, depending on one's point of view.

Unfortunately, this procedure introduces a "splitting error" which typically reduces the time accuracy to first order. The error has been analyzed by Orszag *et al.* [5] and shown to be associated with the formation of a spurious boundary layer due to the fact that the boundary conditions in the projection step fix only one of the velocity components on the boundary and not both. The projection step (b) is equivalent to (and in fact is usually solved as) a Poisson equation for the pressure,

$$\nabla\cdot\nabla p^{n+1/2}=\frac{1}{\Delta t}\nabla\cdot\tilde{\mathbf{u}}. \tag{7}$$

The boundary conditions for this equation are implied by Eq. (6). In the case of time-independent boundary velocities $\mathbf{w}$, both $\mathbf{u}^{n+1}=\mathbf{w}$ and $\tilde{\mathbf{u}}=\mathbf{w}$, and therefore we have $\nabla p^{n+1/2}=0$ on the boundary. Note, however, that this overspecifies the problem because only one component of the pressure gradient on the boundary is sufficient to determine the pressure uniquely. A similar difficulty is also present in the original Navier–Stokes equations (1) and it has been resolved in favor of the pressure gradient component normal to the boundary [6]: a Neumann boundary condition

problem for the pressure. Actually, aside from a question at the initial time, $t \to 0$, the Neumann boundary condition is compatible with both the normal and tangential velocity boundary conditions in the original differential equation system [6]. This is no longer true when the split projection step (b) is employed in the semi-discrete problem, and specifying a Neumann boundary condition for the pressure to satisfy the boundary condition for the normal velocity fails to satisfy the tangential velocity boundary condition and gives rise to the "splitting error" in the form of a spurious boundary layer. Gresho [4] has suggested that some form of this error is inevitable in splitting methods. The object is to minimize the error, which we take to be equivalent to making the numerical method at least second-order accurate in time.

There have been many proposals to restore second-order accuracy by means of modified boundary conditions. These include modifying velocity boundary conditions for the momentum equation [7–9], modifying the homogeneous Neumann boundary condition for the pressure Poisson equation [5, 10], or both [4]. However, using modified boundary conditions is a delicate matter. Some of these schemes work very well (as demonstrated later) but care must be used both in the implementation, since second-order accuracy can be easily destroyed, and in employing the right formulation, since weak instabilities may be induced [5]. Interestingly, there also exist several splitting-type methods for the incompressible Navier–Stokes equations which achieve second-order accuracy without the explicit use of modified boundary conditions. These methods include the pressure correction method of van Kan [11], the incomplete iteration method of Bell *et al.* [12], and Gresho's (simpler) projection 2 [4]. Many of the currently existing methods have been systematized by Gresho [4] from the projection method point of view.

There is an alternative and general approach to operator splitting, called the approximate factorization method [3, 13, 14], that applies to hyperbolic, parabolic, and mixed hyperbolic/parabolic equation systems, usually via an ADI-type splitting. A particular method, called the stabilization method by Marchuk [14] and the "delta" formulation by Warming and Beam [13], is briefly reviewed in Appendix A. This method has the remarkable property that if the underlying (unsplit) fully-discrete equation system is second-order accurate in time, then the split system is also second-order accurate. Questions of intermediate boundary conditions did arise initially [15] because the earlier formulations of the method [13, 16] were put in terms of a semi-discrete formulation, just like Eqs. (2)–(7), in which the question of boundary conditions at intermediate steps was relevant. However, more recent treatments of the method [14] have made it clear that boundary conditions are not an issue as far as second-order accuracy is concerned because the splitting is performed in matrix form, that is, only after the equations have been spatially discretized and the boundary conditions applied. The splitting and the preservation of second-order accuracy is associated with the approximate factorization of the system matrix and not its modification by boundary conditions. This method, therefore, appears to have a number of highly desirable properties. Unfortunately, this type of method has been deemed inapplicable to the incompressible Navier–Stokes equations [8, 16] because the absence of a time derivative term in the continuity equation renders the equations of mixed parabolic/elliptic type. Therefore, to our knowledge, the method has never been applied to the incompressible Navier–Stokes equation system.

In this paper we approach the problem of splitting the Crank–Nicolson discretization of the incompressible Navier–Stokes equations, Eq. (4), from the point of view of the approximate factorization method. We find, in contradiction to existing opinion, that there indeed does exist an approximate factorization associated with these equations that preserves second-order accuracy. In fact, it turns out that those methods that do not require modified boundary conditions [11, 12, 4], mentioned above, are special cases of this approximate factorization. We then generalize the method in two directions. First, we generalize the structure of the approximate factorization so that it becomes a discretization method capable of generating brand-new applications. We illustrate this new capability by showing how to generate second-order accurate splittings for finite-element discretizations with a consistent mass matrix and for methods with a projection step where the continuity constraint is only approximately satisfied. Second, we generalize to the case of an arbitrary order of accuracy.

2. APPROXIMATE SECOND-ORDER FACTORIZATION

For the sake of simplicity we will henceforth restrict ourselves to the Stokes equations, which are sufficient to illustrate the essential features of our method. The Crank–Nicolson form of the Stokes equations may be written as

$$\begin{aligned} \mathbf{u}^{n+1} - \mathbf{u}^n &= -\frac{\Delta t}{2}\nabla(p^{n+1} + p^n) \\ &\quad + \frac{\Delta t}{2}\nu\nabla\cdot\nabla(\mathbf{u}^{n+1} + \mathbf{u}^n), \qquad (8) \\ \nabla\cdot\mathbf{u}^{n+1} &= 0, \end{aligned}$$

together with the appropriate initial and boundary conditions (which are the same as for the Navier–Stokes equations). Note that the pressure gradient term is written in an alternative but equivalent form and that the spatial operators are still in differential form, so that the boundary

conditions are required to be stated separately. It is now convenient to introduce some arbitrary spatial discretization which in general will put the equations in the form

$$\begin{aligned}\mathbf{M}(\mathbf{u}^{n+1}-\mathbf{u}^n)-\frac{\Delta t}{2}\nu\mathbf{L}(\mathbf{u}^{n+1}+\mathbf{u}^n)+\frac{\Delta t}{2}\mathbf{G}(p^{n+1}+p^n)\\ =\Delta t\mathbf{f}^{n+1/2},\\ \mathbf{G}^{\mathrm{T}}\mathbf{u}^{n+1}=g^{n+1},\end{aligned}\tag{9}$$

where $\mathbf{L}$, $\mathbf{G}$, and $\mathbf{G}^{\mathrm{T}}$ are the discrete matrix versions of the Laplacian, gradient, and divergence operators, respectively, each already incorporating the boundary conditions, and $\mathbf{f}$ and g are terms that also arise due to the application of boundary conditions. This is similar to the notation employed by Gresho and Chan [17]; the boundary condition terms should be such as to preserve the second-order accuracy of the discretization, namely, $\mathbf{f}^{n+1/2}$ is time centered and g^{n+1} is defined at the new time. The divergence operator will typically be the matrix transpose of the gradient operator. The matrix operators $\mathbf{M}$ and $\mathbf{L}$ are block diagonal, each block corresponding to a component of velocity. The matrix $\mathbf{M}$ represents the mass matrix in the case of a finite-element discretization, a diagonal matrix in the case of a lumped-mass approximation, and the identity matrix in the case of a finite-difference discretization. For the moment we will restrict ourselves to those spatial discretizations for which $\mathbf{M}=\mathbf{I}$, the identity matrix. We will return to the more general equation later. The above equation may then be rewritten in matrix format, suggestive of Eq. (A1) of Appendix A,

$$\begin{bmatrix}\mathbf{I}-\frac{\Delta t}{2}\nu\mathbf{L} & \frac{\Delta t}{2}\mathbf{G}\\ \frac{\Delta t}{2}\mathbf{G}^{\mathrm{T}} & 0\end{bmatrix}\begin{bmatrix}\mathbf{u}^{n+1}-\mathbf{u}^n\\ p^{n+1}-p^n\end{bmatrix}=\Delta t\begin{bmatrix}\nu\mathbf{L} & -\mathbf{G}\\ -\frac{1}{2}\mathbf{G}^{\mathrm{T}} & 0\end{bmatrix}\begin{bmatrix}\mathbf{u}^n\\ p^n\end{bmatrix}+\Delta t\begin{bmatrix}\mathbf{f}^{n+1/2}\\ \frac{1}{2}g^{n+1}\end{bmatrix},\tag{10}$$

which clearly indicates the structure of the Crank–Nicolson Stokes problem, suggestive of a constrained minimization problem using Lagrange multipliers. The matrix on the left-hand side is typically invertible (aside from details connected with fixing the level of pressure) but its large size is an indication of the difficulty of the fully-coupled implicit Stokes or Navier–Stokes problem. We have chosen to write the equation in this form to emphasize the fact that usually this matrix is symmetric, although this is not necessarily always true. This equation is assumed to be second-order accurate in time. Our object is to find an approximate factorization of the matrix on the left-hand side that preserves this order of accuracy.

We now observe that this equation may indeed be approximately factorized in a manner similar to Eq. (A2) of Appendix A:

$$\begin{bmatrix}\mathbf{I}-\frac{\Delta t}{2}\nu\mathbf{L} & 0\\ 0 & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{I} & \frac{\Delta t}{2}\mathbf{G}\\ \frac{\Delta t}{2}\mathbf{G}^{\mathrm{T}} & 0\end{bmatrix}\begin{bmatrix}\mathbf{u}^{n+1}-\mathbf{u}^n\\ p^{n+1}-p^n\end{bmatrix}=\Delta t\begin{bmatrix}\nu\mathbf{L} & -\mathbf{G}\\ -\frac{1}{2}\mathbf{G}^{\mathrm{T}} & 0\end{bmatrix}\begin{bmatrix}\mathbf{u}^n\\ p^n\end{bmatrix}+\Delta t\begin{bmatrix}\mathbf{f}^{n+1/2}\\ \frac{1}{2}g^{n+1}\end{bmatrix}.$$

Expanding, this equation is equivalent to

$$\begin{aligned}\mathbf{u}^{n+1}-\mathbf{u}^n-\frac{\Delta t}{2}\nu\mathbf{L}(\mathbf{u}^{n+1}+\mathbf{u}^n)+\frac{\Delta t}{2}\mathbf{G}(p^{n+1}+p^n)\\ =\Delta t\mathbf{f}^{n+1/2}+\frac{1}{4}\nu\,\Delta t^2\mathbf{L}\mathbf{G}(p^{n+1}-p^n),\\ \mathbf{G}^{\mathrm{T}}\mathbf{u}^{n+1}=g^{n+1}.\end{aligned}\tag{11}$$

Note that this differs from Eq. (9) by a single additional term on the right-hand side. This term arises naturally as a consequence of the approximate factorization, just as a similar term arises in the same way in Appendix A. It is important to observe that this additional term is $O(\Delta t^3)$, the same order of magnitude as the leading truncation error term of the Crank–Nicolson equation (8), so that the results of using this equation will differ from Crank–Nicolson results by $O(\Delta t^2)$ globally. This equation, therefore, is also a second-order accurate approximation to the Stokes equations, just like the Crank–Nicolson equation. This is precisely analogous to the idea of the approximate factorization method, reviewed in Appendix A, except that now it is applied to the incompressible flow equations rather than to a system of parabolic or hyperbolic equations. As remarked in the Introduction, this was previously considered not possible because of the semi-elliptic nature of the incompressible Navier–Stokes equations [8, 16].

The approximate factorization is equivalent to the following splitting scheme:

$$\text{(a)}\qquad \hat{\mathbf{u}}=\mathbf{u}^n-\frac{\Delta t}{2}\mathbf{G}p^n,$$

$$\left(\mathbf{I}-\frac{\Delta t}{2}\nu\mathbf{L}\right)\tilde{\mathbf{u}}=\left(\mathbf{I}+\frac{\Delta t}{2}\nu\mathbf{L}\right)\hat{\mathbf{u}}+\Delta t\mathbf{f}^{n+1/2},\tag{12a}$$

$$\text{(b)}\qquad \mathbf{u}^{n+1}=\tilde{\mathbf{u}}-\frac{\Delta t}{2}\mathbf{G}p^{n+1},$$

$$\mathbf{G}^{\mathrm{T}}\mathbf{u}^{n+1}=g^{n+1}.\tag{12b}$$

This is analogous to the splitting of Eqs. (5), (6) except that here the discrete Crank–Nicolson operators are being

used and, therefore, there is no question of intermediate boundary conditions. The boundary conditions have already been applied at the level of Eq. (9).

The projection step, Eqs. (12b), is of course first solved for the pressure from the equation

$$\frac{\Delta t}{2}\mathbf{G}^{\mathrm{T}}\mathbf{G}p^{n+1}=\mathbf{G}^{\mathrm{T}}\tilde{\mathbf{u}}-g^{n+1}, \tag{13}$$

where $\mathbf{G}^{\mathrm{T}}\mathbf{G}$ is assumed to be invertible, and then for the velocity from

$$\mathbf{u}^{n+1}=\tilde{\mathbf{u}}-\frac{\Delta t}{2}\mathbf{G}p^{n+1}.$$

To complete the description of the method we must describe the procedure to obtain the initial pressure. Given the initial solenoidal velocity field $\mathbf{u}^0$, the initial pressure p^0 may be obtained by, in effect, solving the Crank–Nicolson equations, or else Eqs. (11), at the initial time in the limit $\Delta t \to 0$. The appropriate equation becomes

$$\mathbf{G}^{\mathrm{T}}\mathbf{G}p^0=\nu\mathbf{G}^{\mathrm{T}}\mathbf{L}\mathbf{u}^0+\mathbf{G}^{\mathrm{T}}\mathbf{f}^0. \tag{14}$$

This splitting of the Crank–Nicolson Stokes equations is, therefore, summarized by Eqs. (12) and (14).

We now remark that Eq. (11), which is the equation effectively solved by the splitting, is basically the same as the equations effectively solved by the splittings of the pressure correction method of van Kan [11], the incomplete iteration method of Bell *et al.* [12], and Gresho's (simpler) projection 2 [4]. These methods, therefore, are basically equivalent to the above realization of the approximate factorization method as far as the splitting scheme is concerned. For example, the splitting (12) may be written in various different but equivalent forms, such as

$$\text{(a)}\quad \left(\mathbf{I}-\frac{\Delta t}{2}\nu\mathbf{L}\right)\tilde{\mathbf{u}}=\left(\mathbf{I}+\frac{\Delta t}{2}\nu\mathbf{L}\right)\mathbf{u}^n-\Delta t\mathbf{G}p^n+\Delta t\mathbf{f}^{n+1/2},$$

$$\text{(b)}\quad \mathbf{u}^{n+1}=\tilde{\mathbf{u}}-\frac{\Delta t}{2}\mathbf{G}(p^{n+1}-p^n),$$

$$\mathbf{G}^{\mathrm{T}}\mathbf{u}^{n+1}=g^{n+1},$$

which corresponds to van Kan's pressure correction method.

3. SPLITTING WITH PRESSURE BOUNDARY CONDITIONS

Before proceeding with the generalizations, we will now describe an existing splitting method based on modified pressure boundary conditions [5] for comparison with the previous results. Again, we deal with a Crank–Nicolson-type discretization and the Stokes equations for simplicity. The splitting takes the form:

(1) Projection equations,

$$\begin{aligned}\tilde{\mathbf{u}}-\mathbf{u}^n&=-\Delta t\nabla p^{n+1/2},\\ \nabla\cdot\tilde{\mathbf{u}}&=0;\end{aligned} \tag{15a}$$

(2) Viscous equations,

$$\frac{\mathbf{u}^{n+1}-\tilde{\mathbf{u}}}{\Delta t}=\frac{1}{2}\nu\nabla\cdot\nabla(\tilde{\mathbf{u}}+\mathbf{u}^{n+1}), \tag{15b}$$

for which the boundary conditions are $\mathbf{u}^{n+1}=\mathbf{w}$, the specified boundary velocities. The reversed order of the steps is inconsequential, since the initial velocity profile must be projected in any case; we are merely following the convention of [5], since we are describing their method. With this ordering one identifies u^{n+1} with the former intermediate velocity and vice versa, and so one accepts u^{n+1} with $\nabla\cdot u^{n+1}\neq 0$ but with the correct boundary conditions, whereas with the former ordering one obtains u^{n+1} with $\nabla\cdot u^{n+1}=0$, but with incorrect boundary conditions. The computed results are identical.

The projection equations are themselves solved in two steps. First, a Poisson equation for the pressure is solved,

$$\nabla\cdot\nabla p^{n+1/2}=\frac{1}{\Delta t}\nabla\cdot\mathbf{u}^n,$$

for which we will consider three types of boundary conditions:

(a) $\mathbf{n}\cdot\nabla p^{n+1/2}=0$, where $\mathbf{n}$ is a unit normal vector at the boundary. This is the "standard" case, discussed in the Introduction, known to be only first-order accurate in time. Note that this produces a velocity at the boundary from Eq. (15a) that is consistent with the specified normal component of the boundary velocity.

(b) $\mathbf{n}\cdot\nabla p^{n+1/2}=-\nu\mathbf{n}\cdot\nabla\times\nabla\times\mathbf{u}^n$. This is the improved pressure boundary condition of Orszag *et al.* [5] designed to produce second-order accuracy. Note that this produces intermediate normal velocities at the boundary that are not consistent with the velocity boundary conditions but, on the other hand, it produces a pressure gradient that is more consistent with the momentum equation at the boundary.

(c) $\mathbf{n}\cdot\nabla p^{n+1/2}=\nu\mathbf{n}\cdot\nabla\cdot\nabla\mathbf{u}^n$. This is very closely related to boundary condition (b). Note that boundary condition (b) is derived from this condition by the use of the vector identity $\nabla^2\mathbf{u}=\nabla(\nabla\cdot\mathbf{u})-\nabla\times\nabla\times\mathbf{u}$ and the assumption of incompressibility. This seemingly equivalent boundary condition [5, 10] is included to illustrate the deleterious effect of even minor variations.

The second step involves the update of the intermediate velocity field $\tilde{\mathbf{u}} = \mathbf{u}^n - \Delta t \nabla p^{n+1/2}$, where a "boundary condition," $\nabla \times \tilde{\mathbf{u}} = \nabla \times \mathbf{u}^n$, is used to extrapolate the intermediate velocity field $\tilde{\mathbf{u}}$ for use in the viscous equations, Eq. (15b). Higher order versions of this method with appropriate extrapolations of boundary condition (b) are described in [18].

4. COMPUTATIONAL EXAMPLES

As a concrete illustration of the foregoing methods we have chosen to implement the MAC staggered-mesh spatial discretization [1] for the Stokes equations with kinematic viscosity $\nu = 0.1$ and to apply it to a series of test problems which are related to one introduced previously [5, 10].

The test problems share the same mesh and boundary conditions and differ in the initial velocity distribution. The problem domain consists of a square box such that $|x|, |y| \leq 1$. The upper and lower boundaries are periodic [illegible] equal to zero at both, and the left and right boundaries are assumed to be no-slip walls ($u, v = 0$). Note that since we have chosen the upper and lower boundaries to be antinodes for the vertical velocity, they are nodes for the pressure and therefore require Dirichlet boundary conditions for the pressure. The mesh consists of 13×13 cells. This coarse discretization is sufficient to resolve the pressure and velocity profiles employed and, in any case, the spatial discretization employed is not important, since we are only interested in questions arising from the time discretization.

We employ three different initial velocity profiles. The u- and v-components at cell faces are defined analytically to satisfy the boundary conditions but they may not satisfy the discrete solenoidal condition. This preliminary velocity field is then projected (in the sense of Chorin) to obtain the initial velocity conditions. The three preliminary velocity fields are:

$$(1)\quad u = \cos\frac{\pi}{2}y\left[\cosh\frac{\pi}{2}x - \frac{\cosh(\pi/2)}{\cos\mu}\cos\mu x\right],$$

$$v = -\sin\frac{\pi}{2}y\left[\sinh\frac{\pi}{2}x - \frac{\sinh(\pi/2)}{\sin\mu}\sin\mu x\right],$$

where $\mu = 2.642442$ is the lowest eigenvalue of the equation $\mu \tan\mu = -(\pi/2)\tanh(\pi/2)$ [5],

$$(2)\quad u = \frac{\pi}{2}\cos\frac{\pi}{2}y[1-x^2]^2,$$

$$v = 4x\sin\frac{\pi}{2}y[1-x^2],$$

$$(3)\quad u = 0,$$

$$v = -\cos\pi y \sin\pi x.$$

Case (1) is the lowest symmetric eigenmode of the analytic test problem treated by Orszag *et al.* (OID) [5], corresponding to a half-period in the vertical direction; however, it is not an exact eigenmode of the discrete problem. Case (2) is a closely related velocity field that does not satisfy the compatibility conditions for the initial velocity field discussed by Deville *et al.* [10], whereas

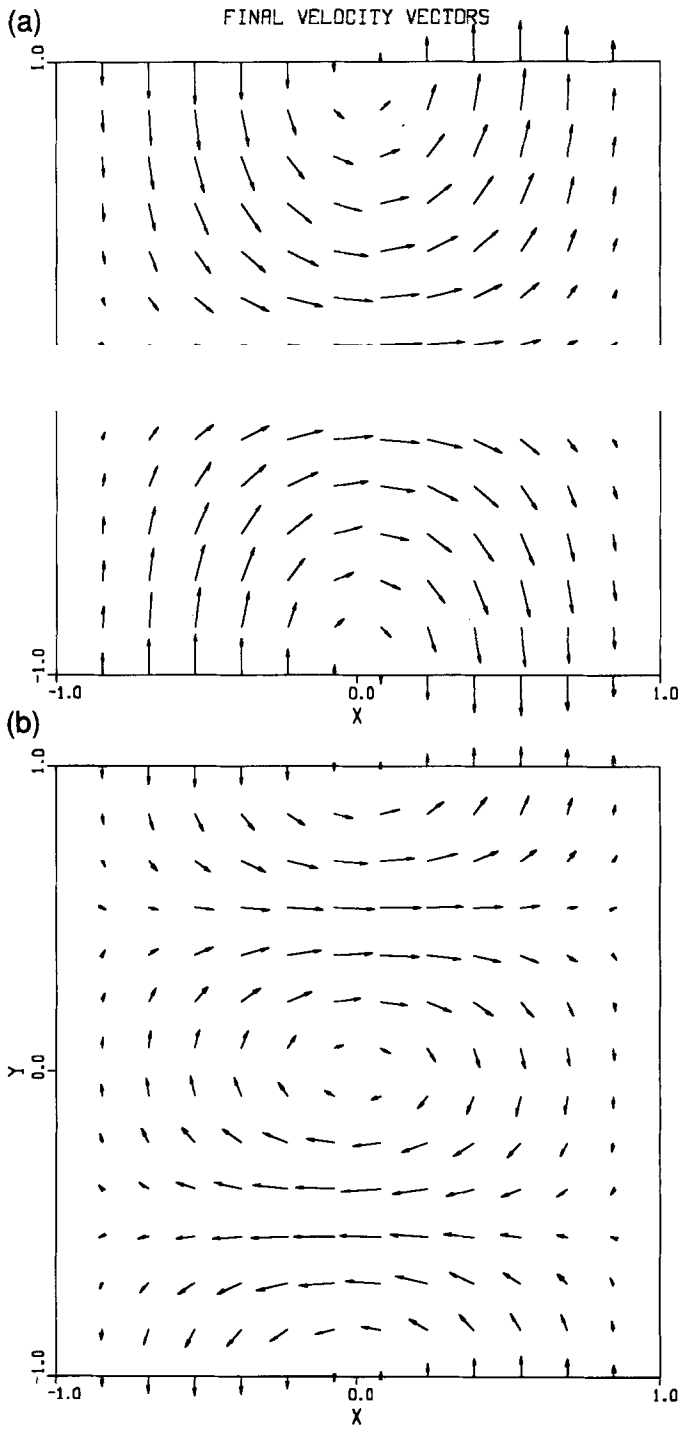

FIG. 1. (a) Final velocity field for cases (1) and (2). (b) Final velocity field for case (3).

case (1) does. In practice, we have not observed any differences in behavior between these two cases, since both are dominated by the lowest symmetric eigenmode of the problem, but we nevertheless retain the latter case for completeness. Both these cases are analytically solenoidal. Case (3) is not initially solenoidal and it also differs in that it represents a full period in the vertical direction.

For each of these cases we have computed the evolution of the velocity profile from the initial time $t=0$ to a final time $t=1$ with time steps Δt differing by a power of 2 for each of the methods: Crank–Nicolson (CN), splitting with pressure boundary conditions (a), (b), (c) of Section (3) (OIDa, OIDb, OIDc, respectively), and the present method (DD). The time steps ranged from $\Delta t = 2^{-7}$ to $\Delta t = 2^{-2}$ in six increments. As a reference for each of the cases we computed the velocity field at the final time using Crank–Nicolson with a very small time step $\Delta t = 2^{-9} = 0.00195$ (i.e., 512 time steps). The solution in each case consists of a decaying velocity field; however, we did not measure the decay rate but instead we monitored the normalized L_2 error:

$$\text{Error} = \frac{\sqrt{\sum |\mathbf{u}_{t=1} - \mathbf{u}_{\text{ref}}|^2}}{\sqrt{\sum |\mathbf{u}_{\text{ref}}|^2}}.$$

This is because much of the error appears as an erroneous velocity profile, in the form of a spurious velocity boundary layer [5], which may not be reflected in the behaviour of the decay rate. Representative velocity fields at time $t=1$ are shown in Fig. 1a for cases (1) and (2), and in Fig. 1b for case (3).

The pressure boundary conditions of Section 3 are quite obvious for the MAC discretization, except for case (c). For this case the boundary condition for the left and right boundaries is

$$\frac{\partial p}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2},$$

and the second derivative of velocity must be extrapolated from the interior. Taking into account the fact that $u=0$ and $\partial u/\partial x = 0$ (from the divergence condition), we obtain at the left boundary, for example,

$$\left[\frac{\partial^2 u}{\partial x^2}\right]_{i=0} = \frac{1}{2\,\Delta x^2}(8u_{i=1} - u_{i=2}),$$

where the subscript i refers to the cell face index in the x-direction, and Δx is the corresponding cell spacing.

The convergence rates of the error for the different

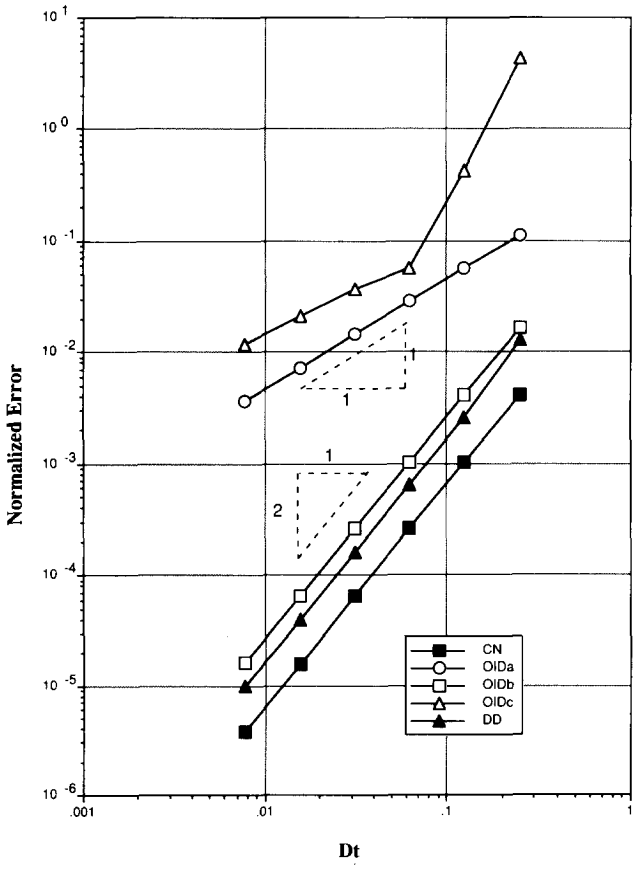

FIG. 2. Error convergence rate for different methods for the case of initial velocity field (1).

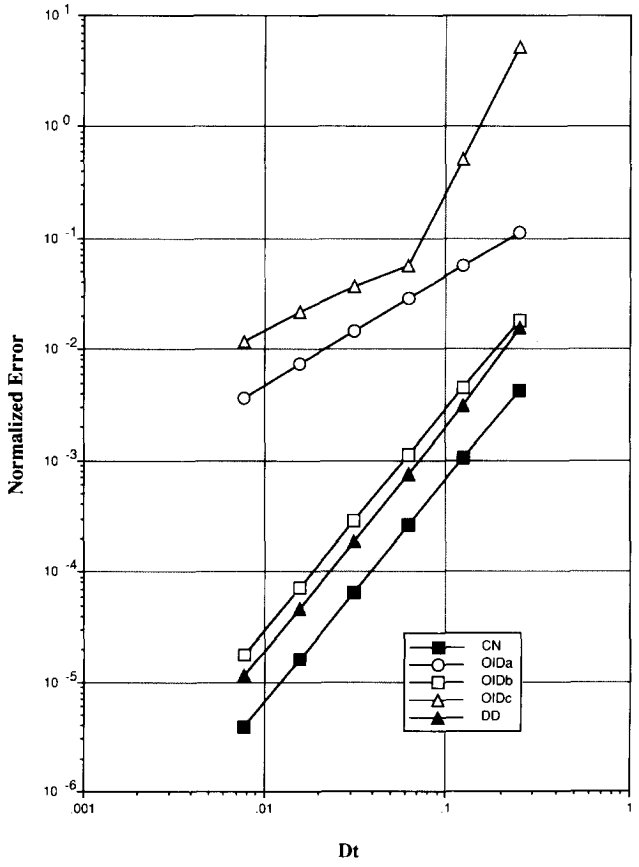

FIG. 3. Error convergence rate for different methods for the case of initial velocity field (2).

methods are shown in Figs. 2, 3, and 4, for the three cases of initial velocity profiles described above. It may be observed in all cases that Crank–Nicolson, the OID improved boundary condition (b), and the present method all display second-order convergence; the standard pressure boundary condition (OIDa) is first-order, and the more conventional modified pressure boundary condition (OIDc) is the worst of all, appearing to only marginally reach asymptotic first-order behavior in the range of time steps investigated. Further, the magnitude of the error is uniformly lower for the second-order methods as compared to the first-order methods. The error of the present method is lowest of the split methods, being only slightly greater than the unsplit Crank–Nicolson error (in fact, the two errors essentially coincide for the third test problem). This comment applies in principle to the other methods [11, 12, 4] that possess the same generic splitting.

Although the behavior of the improved pressure boundary condition (OIDb) of [5] is very satisfactory, it appears that the ostensibly minor variation represented by OIDc not only destroys the order of approximation but it also introduces substantial error, and this is a cautionary note regarding the application of intermediate boundary conditions to split operator equations.

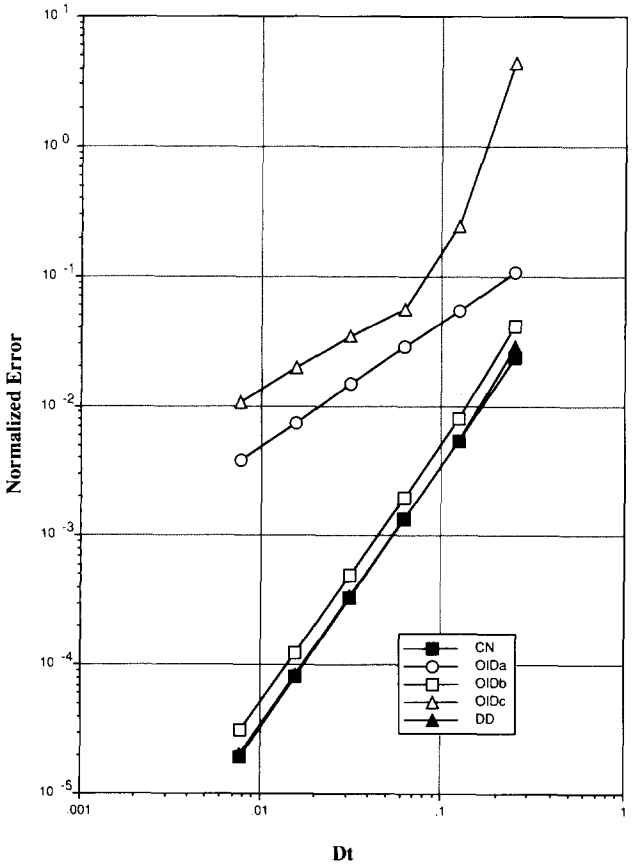

FIG. 4. Error convergence rate for different methods for the case of initial velocity field (3).

5. GENERALIZATION OF THE SECOND-ORDER METHOD AND ITS APPLICATIONS

We return now to consider a discretized equation system which is slightly more general than Eq. (9), which we write as

$$\mathbf{M}(\mathbf{u}^{n+1}-\mathbf{u}^n)-\frac{\Delta t}{2}\mathbf{B}(\mathbf{u}^{n+1}+\mathbf{u}^n)+\frac{\Delta t}{2}\mathbf{G}(p^{n+1}+p^n)$$
$$=\Delta t\,\mathbf{a}^{n+1/2}, \tag{16a}$$
$$\mathbf{G}^{\mathrm{T}}\mathbf{u}^{n+1}=b^{n+1}, \tag{16b}$$

to emphasize its second-order accuracy. Alternatively, we may write it in matrix form as

$$\begin{bmatrix}\mathbf{M}-\frac{\Delta t}{2}\mathbf{B} & \frac{\Delta t}{2}\mathbf{G}\\ \frac{\Delta t}{2}\mathbf{G}^{\mathrm{T}} & 0\end{bmatrix}\begin{bmatrix}\mathbf{u}^{n+1}\\ \Delta p^{n+1}\end{bmatrix}$$
$$=\begin{bmatrix}\mathbf{M}+\frac{\Delta t}{2}\mathbf{B} & -\Delta t\mathbf{G}\\ 0 & 0\end{bmatrix}\begin{bmatrix}\mathbf{u}^n\\ p^n\end{bmatrix}+\Delta t\begin{bmatrix}\mathbf{a}^{n+1/2}\\ \frac{1}{2}b^{n+1}\end{bmatrix}, \tag{17}$$

where $\Delta p^{n+1}=p^{n+1}-p^n$, $\mathbf{M}$ is a positive-definite block diagonal matrix, which may represent the mass matrix associated with a finite-element discretization of the momentum equation or else the identity matrix in the case of a finite-difference discretization, the matrix $\mathbf{B}$ represents those terms in the momentum equation being treated implicitly, such as the viscous terms in Eq. (9) or the Coriolis terms in climate or meteorological models, for example, and $\mathbf{a}^{n+1/2}$ and b^{n+1} represent all the explicitly known terms and boundary condition terms on the right-hand side. Our object is to find a general approximate factorization of the matrix on the left-hand side that preserves the second-order accuracy of the system.

Consider the approximate factorization

$$\begin{bmatrix}\mathbf{M}-\frac{\Delta t}{2}\mathbf{B} & \frac{\Delta t}{2}\mathbf{G}\\ \frac{\Delta t}{2}\mathbf{G}^{\mathrm{T}} & 0\end{bmatrix}$$
$$\approx\begin{bmatrix}\mathbf{M}-\frac{\Delta t}{2}\mathbf{B} & 0\\ 0 & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{I} & \frac{\Delta t}{2}\mathbf{M}^{-1}\mathbf{G}\\ \frac{\Delta t}{2}\mathbf{G}^{\mathrm{T}} & -\frac{\Delta t^2}{4}(\mathbf{L}^*-\mathbf{G}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{G})\end{bmatrix}, \tag{18}$$

where $\mathbf{L}^*$ is an invertible matrix that we are free to choose

arbitrarily, but which should be an approximate Laplacian that in some sense approximates or is close to $\mathbf{G}^T\mathbf{M}^{-1}\mathbf{G}$. Using this approximate factorization of the matrix on the left-hand side of Eq. (17) and expanding, we obtain

$$\mathbf{M}(\mathbf{u}^{n+1}-\mathbf{u}^n)-\frac{\Delta t}{2}\mathbf{B}(\mathbf{u}^{n+1}+\mathbf{u}^n)+\frac{\Delta t}{2}\mathbf{G}(p^{n+1}+p^n)$$

$$=\Delta t\mathbf{a}^{n+1/2}+\frac{\Delta t^2}{4}\mathbf{B}\mathbf{M}^{-1}\mathbf{G}(p^{n+1}-p^n), \tag{19a}$$

$$\mathbf{G}^T\mathbf{u}^{n+1}$$

$$=b^{n+1}+\frac{\Delta t}{2}(\mathbf{L}^*-\mathbf{G}^T\mathbf{M}^{-1}\mathbf{G})(p^{n+1}-p^n). \tag{19b}$$

Note that Eq. (19a) differs from (16a) by a term of $O(\Delta t^3)$, and Eq. (19b) differs from (16b) by a term of $O(\Delta t^2)$. The extra term in the momentum equation is of the same order as the truncation error of this equation, as was the case previously and, therefore, does not change the order of approximation of this equation. However, the extra term in the continuity equation cannot be similarly compared to the truncation error, since the continuity equation has no temporal truncation error and, therefore, it is not clear what effect it has on the order of accuracy of the entire system.

[redacted]

for the pressure of $O(\Delta t^2)$, and therefore globally of $O(\Delta t^2)$ and $O(\Delta t)$, respectively. This is precisely the order of the corresponding global errors associated with Eqs. (16) and, therefore, the approximate factorization of Eq. (18) preserves the order of accuracy of Eqs. (16), as desired.

The approximate factorization given by Eq. (18) is equivalent to the following splitting of Eqs. (16):

$$\text{(a)}\quad \left(\mathbf{M}-\frac{\Delta t}{2}\mathbf{B}\right)(\tilde{\mathbf{u}}-\mathbf{u}^n)=\Delta t(\mathbf{B}\mathbf{u}^n-\mathbf{G}p^n+\mathbf{a}^{n+1/2}), \tag{20a}$$

$$\text{(b)}\quad \frac{\Delta t}{2}\mathbf{L}^*(p^{n+1}-p^n)=\mathbf{G}^T\tilde{\mathbf{u}}-b^{bn+1},$$

$$\mathbf{M}(\mathbf{u}^{n+1}-\tilde{\mathbf{u}})=-\frac{\Delta t}{2}\mathbf{G}(p^{n+1}-p^n). \tag{20b}$$

These equations are solved in sequence for $\tilde{\mathbf{u}}$, p^{n+1}, and $\mathbf{u}^{n+1}$. Note that we have not only accomplished a splitting into three smaller decoupled problems, but also that we have replaced a problem for the pressure involving the $\mathbf{G}^T\mathbf{M}^{-1}\mathbf{G}$ matrix by one involving the $\mathbf{L}^*$ matrix, while preserving second-order accuracy.

Applications

The splitting of Eq. (20) is a generalization of Eqs. (12) and (13). It may not be the most general splitting possible; however, it illustrates the usefulness of the approximate factorization approach. We now demonstrate its usefulness by outlining some sample applications.

(a) *Finite elements.* Gresho and Chan [17] have considered splittings of finite-element discretizations that involve the use of the lumped mass matrix in the pressure solution. Their methods are globally first-order accurate for the velocity. The splitting of Eqs. (20), with $\mathbf{B}$ the viscous or Laplacian matrix and $\mathbf{L}^*=\mathbf{G}^T\mathbf{M}_L^{-1}\mathbf{G}$, where $\mathbf{M}_L$ is the lumped mass matrix, produces second-order global accuracy for the velocity at the cost of solving an extra system involving the consistent mass matrix $\mathbf{M}$. This, however, is inexpensive via conjugate gradients with $\mathbf{M}_L$ as a preconditioner.

(b) *Cell-centered finite-differences or equal-order finite-elements.* The use of cell-centered finite-differences for the incompressible Navier–Stokes equations is attractive because of their greater simplicity, but the operator $\mathbf{G}^T\mathbf{G}$, which appears in Eq. (13) and also in other similar splittings, contains a null-space which leads to the so-called "checkerboard" modes in the pressure. A solution is to replace $\mathbf{G}^T\mathbf{G}$ in Eqs. (13), for example, by a null-space-free Laplacian operator $\mathbf{L}^*$ which approximates $\mathbf{G}^T\mathbf{G}$, but this makes the system only first-order accurate in time [19]. In this case, the use of the splitting (20), with $\mathbf{M}=\mathbf{I}$, restores

[redacted]

An analogous situation exists in the case of equal-order finite-element discretizations [20, 21]. In a similar manner, the use of the splitting of Eq. (20) would permit second-order accuracy in this case also.

(c) *Ocean dynamics.* The GFDL model for ocean dynamics, as described by Semtner [22], for example, leads to a system of two-dimensional barotropic equations very similar to Eqs. (16), except that p is the surface pressure, $\mathbf{u}$ is a vertically-averaged horizontal velocity field, $\mathbf{M}$ is a diagonal matrix involving depth, and $\mathbf{B}$ is an antisymmetric matrix representing Coriolis forces. The antisymmetry of $\mathbf{B}$ is inconvenient because the resulting system matrix is nonsymmetric and therefore more difficult to solve by iterative methods as compared to a symmetric system. In addition, a consistent discretization leads to a nine-point operator containing an undesirable null-space, which is avoided by an ad-hoc replacement of this operator by a null-space-free five-point operator. The use of the splitting of Eq. (20) in this case permits a second-order accurate system split into a nonsymmetric but trivially solved Coriolis system (20a) and a symmetric null-space-free system for the surface pressure (20b) using a five-point operator for the matrix $\mathbf{L}^*$.

6. HIGHER ORDER APPROXIMATE FACTORIZATION

We will now indicate how the ideas of Section 2 may be extended to an arbitrarily high order, assuming that the solutions have sufficient smoothness for this to make sense.

There are many different ways to obtain a high-order discretization for the Navier–Stokes equations. To be specific we will follow the procedure of [5, 18], keeping in mind, however, that our method is more generally applicable. We begin by writing the Navier–Stokes equations in the form

$$\frac{\partial \mathbf{u}}{\partial t} = -\mathbf{G}p + \nu\mathbf{L}\mathbf{u} + \mathbf{N}\mathbf{u},$$

$$\mathbf{G}^{\mathrm{T}}\mathbf{u} = 0,$$

where $\mathbf{G}$ represents the gradient operator, $\mathbf{G}^{\mathrm{T}}$ the divergence operator, $\mathbf{L}$ the linear viscous operator, and $\mathbf{N}$ the nonlinear advection operator. The momentum equation is then integrated over a time step Δt, from t^n to t^{n+1}:

$$\mathbf{u}^{n+1} - \mathbf{u}^n = \int_{t^n}^{t^{n+1}} (-\mathbf{G}p + \nu\mathbf{L}\mathbf{u})\, dt + \int_{t^n}^{t^{n+1}} \mathbf{N}\mathbf{u}\, dt.$$

$$\mathbf{G}^{\mathrm{T}}\mathbf{u}^{n+1} = 0.$$

The nonlinear terms are discretized using an explicit rth-order Adams–Bashforth scheme,

$$\int_{t^n}^{t^{n+1}} \mathbf{N}\mathbf{u}\, dt = \Delta t \sum_{k=0}^{r-1} \gamma_k \mathbf{N}\mathbf{u}^{n-k} + O(\Delta t^{r+1}),$$

and the linear terms are discretized using an implicit rth-order Adams–Moulton scheme,

$$\int_{t^n}^{t^{n+1}} (-\mathbf{G}p + \nu\mathbf{L}\mathbf{u})\, dt = \Delta t \sum_{k=0}^{r-1} \beta_k(-\mathbf{G}p^{n+1-k} + \nu\mathbf{L}\mathbf{u}^{n+1-k}) + O(\Delta t^{r+1}),$$

where γ_k and β_k are appropriate coefficients. Note that it is not necessary for our method that the advection terms be differenced explicitly. The discrete equations may now be summarized as

$$\mathbf{u}^{n+1} - \Delta t\beta_0 \nu\mathbf{L}\mathbf{u}^{n+1} + \Delta t\beta_0 \mathbf{G}p^{n+1} = \mathbf{F}^n + O(\Delta t^{r+1}),$$

$$\mathbf{G}^{\mathrm{T}}\mathbf{u}^{n+1} = 0,$$

where

$$\mathbf{F}^n = \mathbf{u}^n + \Delta t \sum_{k=1}^{r-1} \beta_k(-\mathbf{G}p^{n+1-k} + \nu\mathbf{L}\mathbf{u}^{n+1-k}) + \Delta t \sum_{k=0}^{r-1} \gamma_k \mathbf{N}\mathbf{u}^{n-k}$$

collects the explicit terms. If we now apply boundary conditions, the above equations may be written as

$$\mathbf{u}^{n+1} - \Delta t\beta_0 \nu\mathbf{L}\mathbf{u}^{n+1} + \Delta t\beta_0 \mathbf{G}p^{n+1} = \mathbf{F}^n + \Delta t\beta_0 \mathbf{f} + O(\Delta t^{r+1}), \tag{21}$$

$$\mathbf{G}^{\mathrm{T}}\mathbf{u}^{n+1} = g,$$

where the terms $\mathbf{f}$ and g appear because of boundary conditions and where henceforth we interpret $\mathbf{G}$, $\mathbf{G}^{\mathrm{T}}$, $\mathbf{L}$, and $\mathbf{N}$ to be discrete operators suitably modified by the applications of boundary conditions. We assume that we are able to apply the boundary conditions such that Eq. (21) is rth-order accurate. Our object is to demonstrate that we can define a splitting that preserves this order of accuracy. These equations are analogous to Eq. (10) of Section 2, and they exhibit the same structure and computational complexity.

We now modify the above equation by the addition of a term to the right-hand side. This term is

$$\Delta t^2 \beta_0^2 \mathbf{L}\mathbf{G}\, \nabla^{r-1} p^{n+1}, \tag{22}$$

where ∇ is the backward-difference operator [23], such that

$$\nabla p^{n+1} = p^{n+1} - p^n$$

and

$$\nabla^r p^{n+1} = p^{n+1} - \frac{r}{1!} p^n + \frac{r(r-1)}{2!} p^{n-1} - \cdots = p^{n+1} + P_r^n,$$

where

$$P_r^n = -\frac{r}{1!} p^n + \frac{r(r-1)}{2!} p^{n-1} - \cdots = \sum_{k=1}^{r} \alpha_k^r p^{n+1-k},$$

collects the terms involving explicitly known pressures. It may be shown [23] that

$$\nabla^{r-1} p^{n+1} = \Delta t^{r-1} \left[\frac{\partial^{r-1} p}{\partial t^{r-1}}\right]^{n+1} + O(\Delta t^r),$$

and hence the term (22) is $O(\Delta t^{r+1})$, the same order of accuracy as the truncation error of Eq. (21). Supplemented by this term, Eq. (21) may now be factored as follows:

$$\begin{bmatrix} \mathbf{I} - \Delta t\beta_0 \nu\mathbf{L} & 0 \\ 0 & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \Delta t\beta_0 \mathbf{G} \\ \Delta t\beta_0 \mathbf{G}^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}^{n+1} \\ p^{n+1} \end{bmatrix} = \begin{bmatrix} \mathbf{F}^n + \Delta t^2 \beta_0^2 \mathbf{L}\mathbf{G} P_{r-1}^n \\ 0 \end{bmatrix} + \Delta t\beta_0 \begin{bmatrix} \mathbf{f} \\ g \end{bmatrix}.$$

This corresponds to the approximate factorization of Section 2, but now for an rth-order accurate scheme. The splitting equivalent to Eqs. (12) follows immediately.

7. SUMMARY

The methods of approximate factorization are known to be very effective as second-order accurate methods for evolution equations of parabolic or hyperbolic type [13, 14]. We have now shown how to extend the ideas underlying these methods to the equations of incompressible flow. The resulting splitting is found to be closely related to some existing second-order methods whose connection to approximate factorizations was not suspected. The test problems indicate that this type of splitting is very accurate, with an error that is only slightly worse than the unsplit Crank–Nicolson method.

This application of the approximate factorization method thus provides an alternative approach to, as well as new insight into, existing splitting methods for the incompressible Navier–Stokes equations. This is not all, however. The method of approximate factorization is a powerful tool which may be extended to derive brand-new splittings. We have demonstrated two such generalizations. In the one case we have derived a generalized form of a second-order accurate splitting which permits a variety of special cases; we have illustrated its usefulness with three sample applications which by no means exhaust the possibilities. In the other case we have indicated how to extend the original method to higher than second-order accuracy.

APPENDIX A: THE METHOD OF APPROXIMATE FACTORIZATION [13, 14]

Let us assume we have a second-order accurate discretization of an evolution equation or of a system of evolution equations of the form

$$\frac{\phi^{n+1}-\phi^n}{\Delta t}+\frac{1}{2}\mathbf{A}(\phi^{n+1}+\phi^n)=0,$$

where $\mathbf{A}$ represents a discrete operator matrix and ϕ is the vector of unknowns. This equation may alternatively be written as

$$\left(\mathbf{I}+\frac{\Delta t}{2}\mathbf{A}\right)(\phi^{n+1}-\phi^n)+\Delta t\mathbf{A}\phi^n=0. \tag{A1}$$

Now, let us assume that the operator $\mathbf{A}$ may be split as $\mathbf{A}=\mathbf{A}_1+\mathbf{A}_2$, where $\mathbf{A}_1$ and $\mathbf{A}_2$ are nonsingular, easily solvable matrices. The above equation is then approximately factorized as follows:

$$\left(\mathbf{I}+\frac{\Delta t}{2}\mathbf{A}_1\right)\left(\mathbf{I}+\frac{\Delta t}{2}\mathbf{A}_2\right)(\phi^{n+1}-\phi^n)+\Delta t\mathbf{A}\phi^n=0. \tag{A2}$$

Expanding, this equation is equivalent to

$$\left(\mathbf{I}+\frac{\Delta t}{2}\mathbf{A}\right)(\phi^{n+1}-\phi^n)+\Delta t\mathbf{A}\phi^n+\frac{\Delta t^2}{4}\mathbf{A}_1\mathbf{A}_2(\phi^{n+1}-\phi^n)=0.$$

Note that this differs from (A1) by a term of $O(\Delta t^3)$; that is, this equation has global second-order accuracy, just like Eq. (A1). Equation (A2) is, of course, equivalent to the splitting:

$$\text{(a)}\qquad \left(\mathbf{I}+\frac{\Delta t}{2}\mathbf{A}_1\right)\phi^*+\Delta t\mathbf{A}\phi^n=0,$$

$$\text{(b)}\qquad \left(\mathbf{I}+\frac{\Delta t}{2}\mathbf{A}_2\right)(\phi^{n+1}-\phi^n)-\phi^*=0.$$

Note that this is reminiscent of the splittings of Sections (1) and (2); however, the approximate factorization of Eq. (A2) is dependent on the structure of the matrix $\mathbf{I}+\Delta t/2\mathbf{A}$ of Eq. (A1), characteristic of evolution equations of parabolic or hyperbolic type, which differs essentially from the structure of the corresponding matrix of Eq. (10).

APPENDIX B: AN ERROR ANALYSIS OF THE GENERALIZED SPLITTING, EQ. (20)

Consider the second-order accurate Crank–Nicolson discretization given by Eq. (16),

$$\begin{aligned}&\mathbf{M}(\mathbf{u}^{n+1}-\mathbf{u}^n)-\frac{\Delta t}{2}\mathbf{B}(\mathbf{u}^{n+1}+\mathbf{u}^n)+\frac{\Delta t}{2}\mathbf{G}(p^{n+1}+p^n)\\&\qquad=\Delta t\mathbf{a}^{n+1/2}+O(\Delta t^3),\\&\mathbf{G}^{\mathrm{T}}\mathbf{u}^{n+1}=b^{n+1},\end{aligned} \tag{B1}$$

where the error term indicates the order of magnitude of the temporal truncation error. Formally solving this system for $\mathbf{u}^{n+1}$ and p^{n+1} indicates that the truncation error produces errors of $O(\Delta t^3)$ in the velocity and $O(\Delta t^2)$ in the pressure. This means that the global error for the velocity is $O(\Delta t^2)$ and for the pressure it is $O(\Delta t)$.

Now consider the system given by Eq. (19), which is equivalent to the splitting specified by Eq. (20),

$$\begin{aligned}&\mathbf{M}(\mathbf{u}^{n+1}-\mathbf{u}^n)-\frac{\Delta t}{2}\mathbf{B}(\mathbf{u}^{n+1}+\mathbf{u}^n)+\frac{\Delta t}{2}\mathbf{G}(p^{n+1}+p^n)\\&\qquad=\Delta t\mathbf{a}^{n+1/2}+E_1^{n+1/2},\\&\mathbf{G}^{\mathrm{T}}\mathbf{u}^{n+1}=b^{n+1}+\frac{\Delta t}{2}(\mathbf{L}^*-\mathbf{G}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{G})(p^{n+1}-p^n)\\&\qquad=b^{n+1}+E_2^{n+1},\end{aligned} \tag{B2}$$

where $E_1^{n+1/2}=O(\Delta t^3)$ is the combined error in the momentum equation, and $E_2^{n+1}=O(\Delta t^2)$ is the error in the continuity equation. We solve the momentum equation for

the velocity difference $\mathbf{u}^{n+1}-\mathbf{u}^n$ and eliminate it using the continuity equation to obtain

$$\frac{\Delta t}{2}\mathbf{G}^{\mathrm{T}}\left(\mathbf{M}-\frac{\Delta t}{2}\mathbf{B}\right)^{-1}\mathbf{G}(p^{n+1}+p^n)+\Delta t\mathbf{G}^{\mathrm{T}}\left(\mathbf{M}-\frac{\Delta t}{2}\mathbf{B}\right)^{-1}\times(\mathbf{B}\mathbf{u}^n-\mathbf{a}^{n+1/2})+b^{n+1}-b^n$$
$$=\mathbf{G}^{\mathrm{T}}\left(\mathbf{M}-\frac{\Delta t}{2}\mathbf{B}\right)^{-1}E_1^{n+1/2}-(E_2^{n+1}-E_2^n), \qquad \text{(B3)}$$

where, for the case of time-independent operators, we can write

$$E_2^{n+1}-E_2^n=\frac{\Delta t}{2}(\mathbf{L}^*-\mathbf{G}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{G})(p^{n+1}-2p^n+p^{n-1})$$
$$=O(\Delta t^3).$$

Note that all the terms on the left-hand side of (B3) are $O(\Delta t)$, while the two error terms on the right-hand side are both $O(\Delta t^3)$. Thus, the error in the continuity equation E_2 [illegible] solution of the pressure as the momentum equation error E_1. Equation (B3) indicates that the pressure p^{n+1} accumulates errors of $O(\Delta t^2)$. Substituting this in the momentum equation leads to the conclusion that the velocity $\mathbf{u}^{n+1}$ accumulates errors of $O(\Delta t^3)$. This means that the global errors for the velocity and pressure are $O(\Delta t^2)$ and $O(\Delta t)$, respectively, which is exactly the same as for the unsplit system, Eq. (B1).

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